To Ask and to Explain

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1 Understanding our World

Mathematics is the language of the sciences, and for good reason. Many natural phenomena are governed by scientific laws that are often expressed as equations. In this sense, mathematical equations govern *how* things work.

But more fundamental to mathematics is its ability to explain *why* things work. Why does an apple fall to the ground when dropped? This behaviour stems from a law of nature, formulated by Sir Isaac Newton in his monumental book *Mathematical Principles of Natural Philosophy* (often abbreviated as *Principia*[1]), one of the most important scientific works in history (Newton 1687a).

Law of universal gravitation. Any two objects attract each other with a force (called *gravity*) proportional to the masses of both objects. The force also depends on the distance between the centers of the objects. ☆

A mathematical relationship governs *how* they attract: double the mass of either object, double the force. The apple and the planet earth attract each other with a force that draws the apple to the earth. The force also strengthens as the objects come closer, and weakens as they go further apart. Note that the force doesn't care about the type of object; tasty fruit or rocky planet, only how massive they are matters.

A dropped apple begins as stationary but immediately starts to pick up speed; it keeps getting faster until it hits the ground. This speeding-up process is called *acceleration*; we run when we wish to accelerate more (gain more speed) than by walking. When a salesman for a car boasts that it "goes from zero to 80 kph in 2 seconds", he is advertising the car's acceleration.

This article, last updated on 17 May 2014, is part of the first chapter of a math book for general audiences that I am writing, tentatively titled *Connection and Classification*.

[1] The book's original Latin name is *Philosophiæ Naturalis Principia Mathematica*.

(a) (b)

Figure 2 – (a): Portrait of Sir Isaac Newton (1642–1767). (b): Isaac Newton's own annotated copy of *Principia* [1]. Newton had made handwritten corrections on this copy of *Principia* to prepare for its second edition.

Drop an apple and a cannonball from the same height; both accelerate until they strike the ground **at the same time**. In particular, at any point in the falling process the apple and cannonball are always at the same height, and are in fact falling at the same speed; thus they have the same acceleration.

But why should they? If the cannonball is much heavier than the apple, shouldn't it experience a stronger pull towards the earth than the apple, and accelerate faster? After all, when we push something with more force, doesn't it speed up faster? The reason is another law:

Newton's second law. If we wish to move an object with mass m at an acceleration a, we need a force F with a strength of:

$$F = m \times a \qquad ☆$$

Double the mass an object, and it'll take twice the amount of force to move the object at the original acceleration. Basically, it's more difficult to move heavier objects. The greater pull on the cannonball is perfectly counterbalanced by the greater difficulty in moving it!

Now we are ready for a mathematical explanation of *why* the counterbalancing occurs:

Observation. Drop a cannonball and an apple from the same height, and they always fall with the same acceleration. ☆

EXPLANATION The cannonball is understandably rather massive—not double the apple's mass, nor triple, but probably N times the apple's mass for some large number N.

$$\text{cannonball mass} = N \times \text{apple mass} \tag{1}$$

The law of universal gravitation then allows us to compare the attractions between the earth and each object. Note that we can ignore the effect of distance on the attractive forces, because the apple and the cannonball are at nearly the same (huge) distance from the center of the earth.

$$\text{cannonball-earth attraction} = N \times \text{apple-earth attraction} \tag{2}$$

For the rest, we begin with Newton's second law, as applied to both objects:

$$\begin{aligned}
\text{cannonball-earth attraction} &= \text{cannonball mass} \times \text{cannonball acceleration} && (3)\\
\text{apple-earth attraction} &= \text{apple mass} \times \text{apple acceleration} && (4)\\
\implies N \times \text{apple-earth attraction} &= N \times \text{apple mass} \times \text{apple acceleration} && (5)
\end{aligned}$$

(the $\implies$ sign is read as "implies", and it means that the equation after it is a consequence of the equation before it.) We can perform some substitutions using some of our previous equations:

$$\begin{aligned}
\underbrace{N \times \text{apple-earth attraction}}_{\text{Equation (2)}} &= \underbrace{N \times \text{apple mass}}_{\text{Equation (1)}} \times \text{apple acceleration}\\
\implies \text{cannonball-earth attraction} &= \text{cannonball mass} \times \text{apple acceleration} && (6)
\end{aligned}$$

Finally, by comparing with Equation (3),

$$\begin{aligned}
\text{cannonball mass} \times \text{cannonball acceleration} &= \text{cannonball mass} \times \text{apple acceleration} && (7)\\
\implies \text{cannonball acceleration} &= \text{apple acceleration} && (8)
\end{aligned}$$

QED

("QED" stands for the Latin phrase *quod erat demonstrandum*, which signals that "we have achieved what we wanted to demonstrate". It is often used to mark the end of an explanation.)

The flow of the explanation can be visualized as a diagram with arrows:

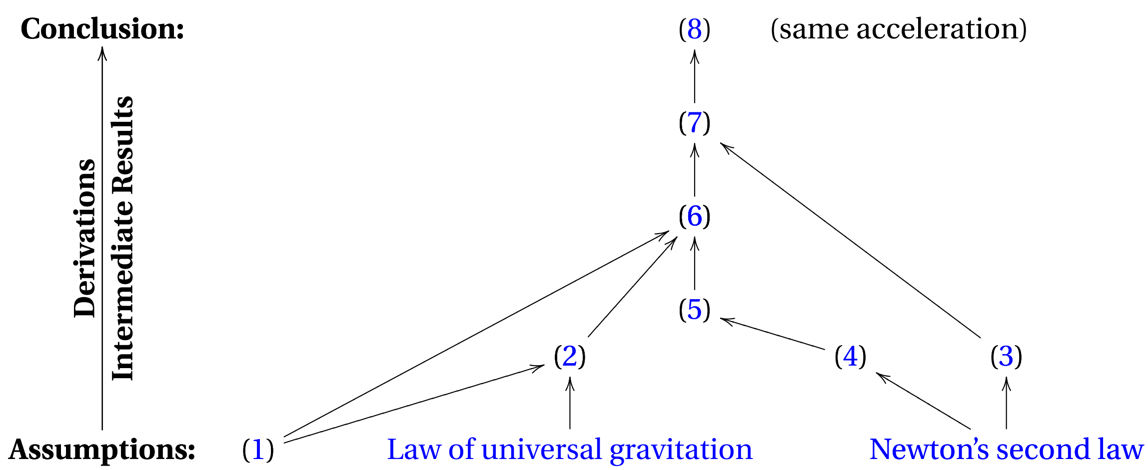

Figure 3 – The anatomy of a mathematical explanation. Each arrow pointing from P to Q means that P is used in the derivation of Q.

And there you have it, a mathematical explanation of *why* a cannonball falls with the same acceleration as a much lighter apple. Mathematical explanations begin with some assumptions, which in our case were the relative masses of the cannonball and the apple (Equation (1)), as well as the law of universal gravitation and Newton's second law. We then chart our course through a network of interlocking logical deductions, climbing a tower of intermediate results—every result standing on the shoulders of the results before it. Each logical step always begins from what is known (assumptions or previously derived results) and moves toward what is not yet certain. One might say that we pull the explanation up by its bootstraps! At last, at the summit of the tower we clinch the conclusion, the destination, the curious thing that we sought to explain. QED.

Provided that our assumptions are correct and every logical step in our derivations is sound, the conclusion (same acceleration) we ultimately derive is beyond dispute. For this reason, mathematical explanations provide watertight evidence, and are usually called *proofs*. The precision and rigor of mathematical proofs have been prized since the ancient Greeks, starting from around the 5$^{\text{th}}$ century BC. They were not the first to formulate equations or systems of calculation, but they were the first to prove their mathematical knowledge. From a set of assumptions, knowledge can be derived and proved, while novel arguments continue to use old knowledge to derive new insights, just like the bootstrapping tower from before. This bloom of knowledge constitutes a theory, an entire world spawned from a few assumptions. A whole universe lies encapsulated within those humble seeds.

The stringent demands of mathematical proof set the bar high for what can be considered as *mathematical truth*. In a way, proven mathematical truth is the surest standard; it has delivered spectacularly as the foundation of all mathematical and scientific knowledge, and it continues to maintain our course on our search for **truth**.

2 Through a Mathematical Lens

Truth is the goal of science; it used to be called *natural philosophy*, that is, thinking about nature. Since the ancient times, people have wondered about the land and the sea, about plants and animals, and about the stars in the heavens. *What* is everything made of? *How* do things work? *Why* do they work that way? "Nature-thinkers" had devised various theories to answer those questions; let's survey the development of *astronomy*, the science of stars and planets.

Around the 5$^{\text{th}}$ century BC, The great Greek mathematician-philosopher Plato tasked the students of astronomy to explain the movements of planets as observed in the sky, but using a mechanism involving uniform circular motions, because that was viewed to be "perfect". Plato wanted a *mathematical explanation* of planetary motion; he yearned to find simple, "perfect", beautiful reasons behind the complicated "dance" of the planets.

Several attempts to find such an explanation culminated in the mathematician-astronomer Claudius Ptolemy's publication of the *Almagest*[2] (2$^{\text{nd}}$ century AD), in which Ptolemy combined some work from his predecessors[3] to model each planet as travelling in a circle called an *epicy-*

[2] *Mathēmatikē Syntaxis* was its original Greek name, meaning "Mathematical Treatise", but later Arabic admirers named it the *al-Majisṭī*, meaning "The Greatest".

[3] Most notably Eudoxus and Hipparchus; I am skipping a lot of history here. A far more thorough history of mathematical astronomy can be found in (Linton 2004).

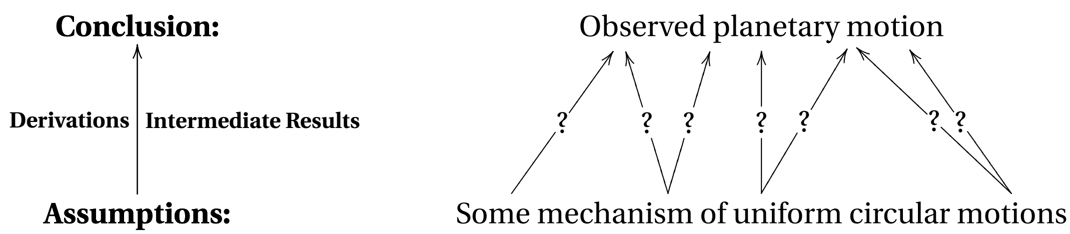

Figure 4 – Plato's task to the students of astronomy.

cle, while the epicycle revolved in another circle called the *deferent*. In order to match astronomical data, the deferent was centered near the earth but not on it, and the circular motions sometimes sped up or slowed down; the exact variation in speed was an original contribution by Ptolemy. The *Almagest* also provided tables of that data so astronomers could use his "mechanism of circles" to calculate the positions of the planets at any time in the future or the past. The "Ptolemaic model" became the authoritative standard in astronomy for over a millenium.

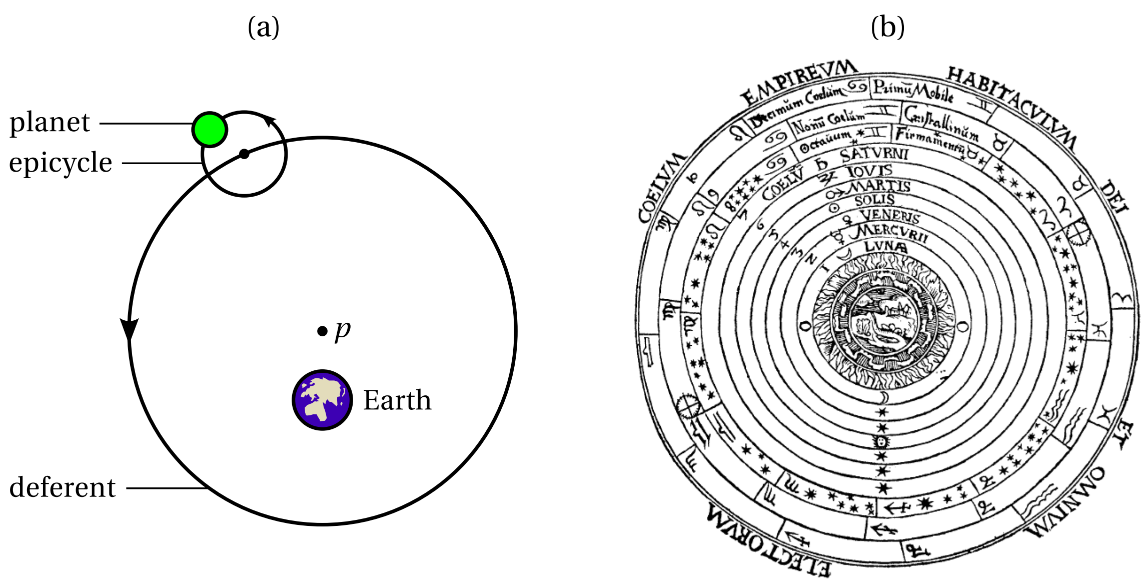

Figure 5 – The Ptolemaic model of planetary motion. (a): The deferent is centered at p, which is close to the Earth. (b): A 16$^{\text{th}}$-century schematic of Ptolemy's ordering of the known planets (those visible with the naked eye), sun and moon, numered as follows: 1. Moon, 2. Mercury, 3. Venus, 4. Sun, 5. Mars, 6. Jupiter, 7. Saturn.

The Ptolemaic model is an example of a *mathematical model*, where real-world phenomena are interpreted mathematically—sometimes using numbers like the coordinates (longtitude and lattitude) of a planet relative to the sky, as well as the date and time of observation—to "transfer" them to the mathematical realm, after which assumptions help to set the stage. Calculations based on those assumptions then derive and explain new mathematical truths, which are interpreted back into the real world to "transfer" a mathematical truth into a real-world conclusion.

Science operates by viewing the world through a mathematical lens.

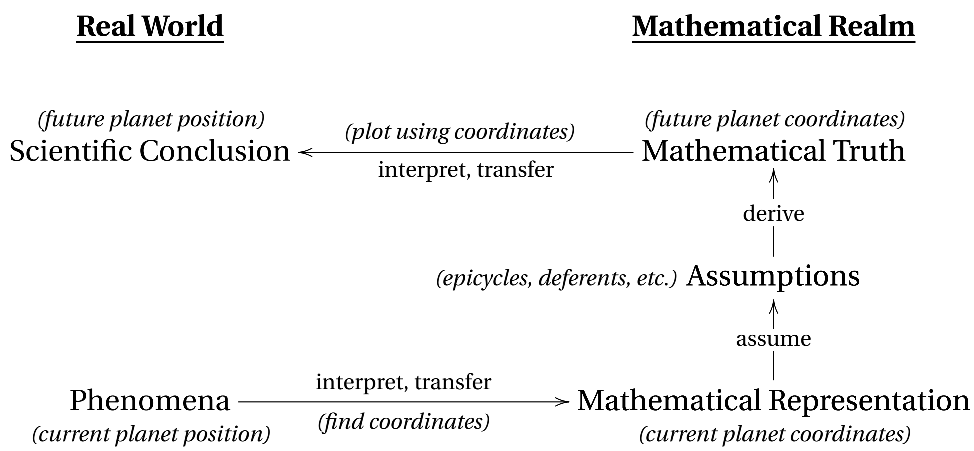

Figure 6 – Mathematical modelling. *(Italicized labels)* illustrate each step of the procedure of using the Ptolemaic model to predict planetary motion.

A mathematical model serves to *explain* and to *predict* phenomena; from the explanation we elucidate the causes and effects (the *whys* and *whats*) that underlie phenomena, and an understanding of that causality gives us the power to predict phenomena.

Is mathematical truth absolute? It is an *absolute consequence* of the assumptions made, but different assumptions can yield different truths and offer novel perspectives. Different mathematical models of the same phenomena, built upon different sets of assumptions, can yield different scientific conclusions; even when the conclusions are the same (e.g. predict roughly the same planet positions), different models still give us new angles to see the world from.

Fast-forward to the Renaissance, when Nicolaus Copernicus shook the assumption of an immobile Earth by arguing that planets seemed to move in strange paths partly because we, the observers, were moving too! Copernicus proposed that the Earth and the other planets should be orbiting (revolving around) the Sun instead (Copernicus 1543), but he also used epicycles and deferents. One of the supporters of his shocking theory was mathematician-astronomer Johannes Kepler, who eventually formulated his own groundbreaking model of planetary motion based on patterns he had noticed in the astronomical data painstakingly recorded by his mentor. The "laws" he gleaned from the data are roughly presented as follows:

Kepler's laws of planetary motion (Kepler 1609, 1619).

1. Planets travel around the Sun in an *ellipse* (circle stretched in one direction). (Figure 7(a))

2. The line joining a planet with the sun sweeps out the same area A within the same time period t. (Figure 7(b))

3. Let T be the amount of time taken by a planet to complete one circuit around the Sun, and let d be the diameter of its ellipse. Then the number T^2/d^3 is the same among all planets. (Figure 7(c)) ☆

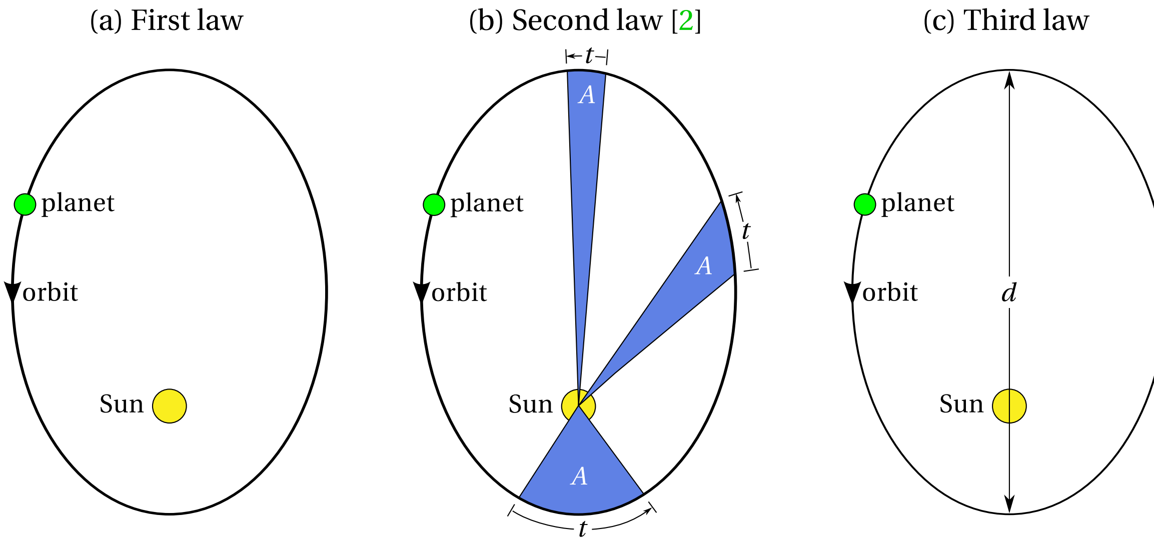

Figure 7 – The Keplerian model of planetary motion. The orbits are exaggerated for clarity; most planets have a nearly circular orbit—a circle that's stretched only a little bit.

Kepler's first law did away with epicycles and deferents; his second law predicts that planets speed up when they are close to the sun but slow down when they are far away (Figure 7(b) shows why); and his third law predicts that planets with larger orbits take longer to travel around them (a larger d needs to be counterbalanced by a larger T). Three simple laws to govern planetary motion!

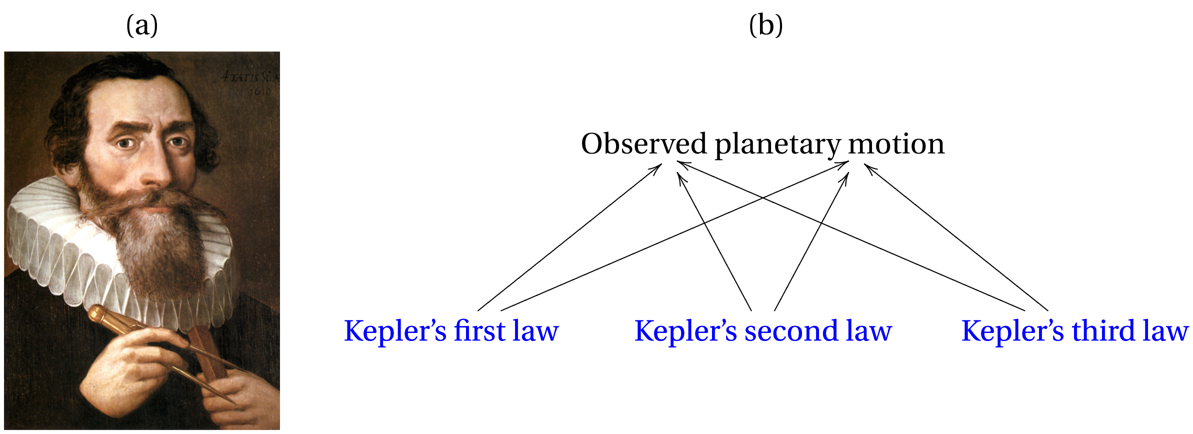

Figure 8 – (a): Portrait of Johannes Kepler (1571–1630). (b): Kepler's laws can be used to predict planetary motion.

Explaining a vast collection of phenomena using just a few laws is very appealing, because it makes us feel that we understand the "inner structure" of the phenomena—*where* they came from, *how* they came to be, and *why* they appear as they are. There is a **unity** in encompassing many phenomena within a few laws; everything simply consists of different physical manifestations of the same few laws. Behind the apparent chaos is a simple underlying **order**, and that order is summarized into laws.

If scientists and mathematicians pursue truth, then we could also say they pursue beauty in the form of simplicity, unity and order.

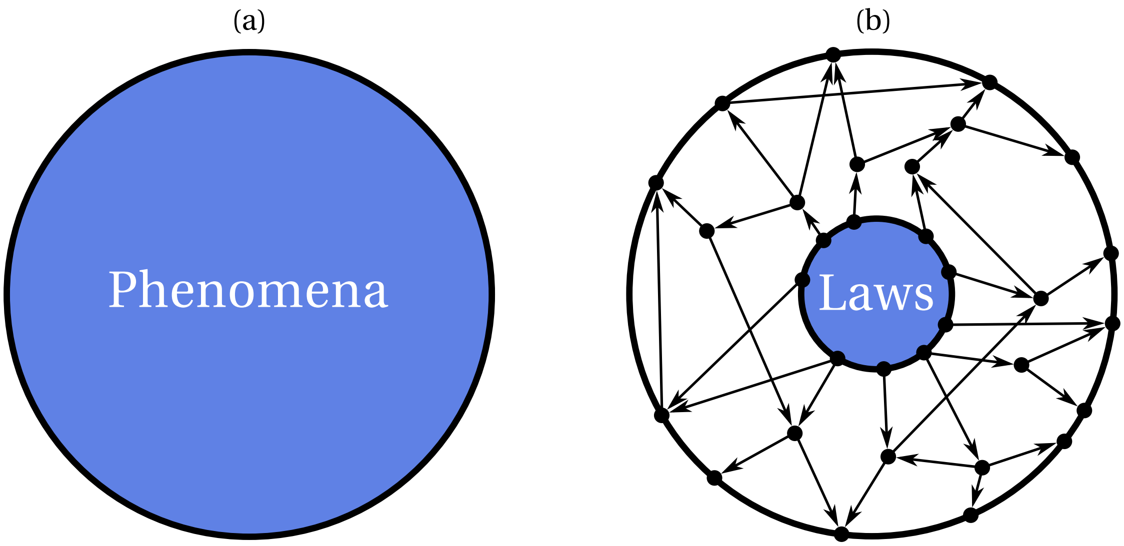

Figure 9 – (a): Phenomena that are not understood. (b): Understanding the "inner structure" of phenomena by deriving them from laws. Black dots represent statements (assumptions, intermediate results or conclusions).

Those elements of beauty were characterized in Sir Isaac Newton's "Rules of Reasoning in Philosophy" listed in his *Principia*:[4]

Newton's rules of reasoning.

1. Don't propose more causes of phenomena than strictly necessary.
2. Therefore, try to explain as many phenomena as possible using the same causes.
3. If all objects seem to have a certain property in every experiment, then propose that the property is universal for all objects. (E.g. the property of mass)
4. If a hypothesis is found to work in every experiment, then it should be accepted as very nearly true—even if it goes against our own reasoning—until some experiment throws an exception. ☆

This encapsulates the "scientific method", the philosophy that guides scientists even today.

Exercise 1 According to Newton's rules of reasoning, should a good mathematical model include assumptions that can be derived (explained, proven) from the other assumptions? *

[4]Based on the translation in (Newton 1687b).

Kepler's illuminating explanation of planetary motion was still wanting, for his laws only applied to planets and held no meaning for whatever happened back on Earth. Newton shattered that divide by deriving Kepler's laws from his own law of universal gravitation and Newton's second law.[5] The latter is actually part of a set of laws proposed in *Principia*:

Newton's laws of motion.

1. Without an external force, an object either remains stationary or travels at constant speed.
2. $F = m \times a$
3. An object that exerts a force on another receives a force of the same magnitude but opposite in direction. (Push on a wall and feel it pushing back.) ☆

These laws explain not only planetary motion but also a gigantic variety of phenomena around us; they have led to a deep mathematical understanding of most of the behaviour of all forms of solids, liquids and gases, and have laid the foundation for much of engineering. The Newtonian model **unified** astronomy and everyday happenings by showing that the movement of the heavens shares the same causes as the phenomena back on Earth, exemplifying Newton's rules of reasoning.

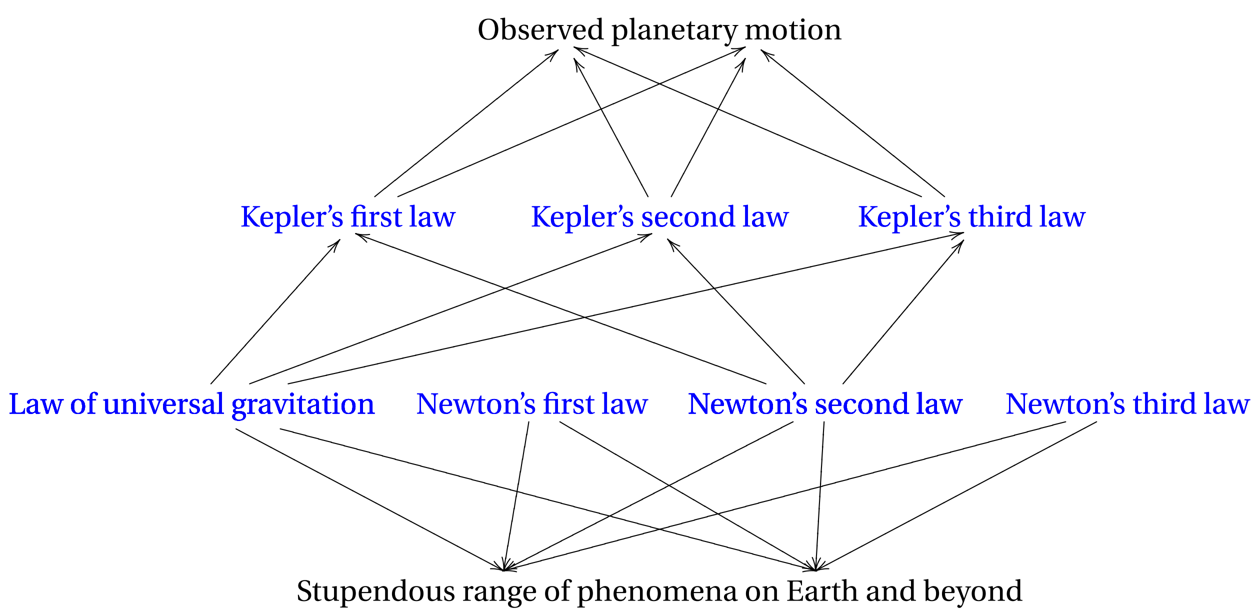

Figure 10 – Newton's laws can be used to derive Kepler's laws, thus predict planetary motion, but also a vast range of other phenomena.

[5]Indeed, the page of *Principia* displayed in Figure 2(b) features part of Newton's derivation of Kepler's second law.

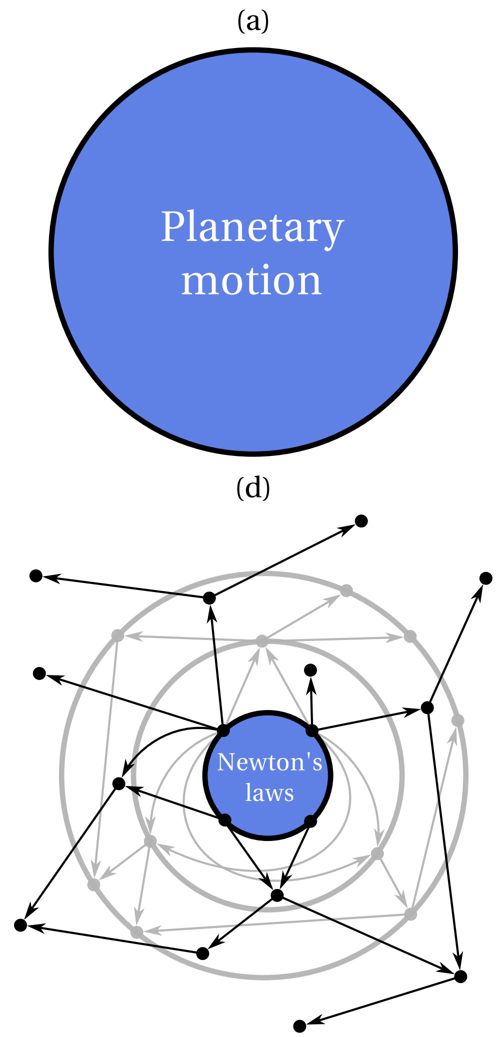

Figure 11 – (a): Planetary motion that is not understood. (b): Explaining planetary motion using Kepler's laws. (c): Explaining Kepler's laws using Newton's laws. (d): Newton's laws explain far more phenomena, in astronomy or outside of it.

Remarks

The scientific consensus is that Newton's laws gave a far better explanation of the natural world than Kepler's laws; in fact we might doubt that Kepler's laws were any explanation at all! One might complain that Kepler's laws simply dictated *how* planets move without explaining *why*, while Newton's laws let us picture the cause and effect as objects pushing on each other or being tugged in certain trajectories. But aren't Newton's laws of motion also rules governing *how* objects move? At what point does a *how* turn into a *why*? Or is the *why* just an illusion? These doubts are worth pondering about.

The Keplerian and Newtonian models are both mathematically sound, and they both rest on assumptions; how are we to judge which is better? The scientific community prefers the Newtonian model, based on Newton's rules of reasoning which it has assimilated. However, after all this talk about beautiful qualities of a mathematical model—such as simplicity, unity and order—we must bear in mind that **truth** is more important; even the model with the simplest laws and most elegant reasoning must be rejected if experiments provide evidence to the contrary. That is the truly fundamental principle in science; but from now on we will consider mathematical theories, derivations and explanations in their own right, freed from a real-world context.

A mathematical explanation is only as strong as its "weakest link", its assumptions. Is it possible to devise a mathematical explanation for which every statement (conclusion, intermediate result, whatever) is solidly justified in terms of other statements? We will address this question shortly, but first let's hear an adorable anecdote related by Stephen Hawking (1988):

> A well-known scientist (some say it was Bertrand Russell) once gave a public lecture on astronomy. He described how the earth orbits around the sun and how the sun, in turn, orbits around the center of a vast collection of stars called our galaxy. At the end of the lecture, a little old lady at the back of the room got up and said: 'What you have told us is rubbish. The world is really a flat plate supported on the back of a giant tortoise.' The scientist gave a superior smile before replying, 'What is the tortoise standing on?' 'You're very clever, young man, very clever,' said the old lady. 'But it's turtles all the way down!'

Figure 12 – It's turtles all the way down!

3 Proof Diagrams

Questions about phenomena and mathematical objects (numbers, equations, shapes etc.) can be answered using mathematical explanations, or proofs. But what about questions concerning proofs themselves? How do we analyze a proof? The trick is to treat a proof itself as a mathematical object, and reason about it mathematically. Write mathematical explanations about proofs!

So far we've accustomed to visualizing a proof using a diagram of statements (assumptions, intermediate results or conclusions) and arrows between them. We even represented statements as black dots in Figure 9 and Figure 11. Let's continue with that style because we're concerned about the "structure of reasoning" itself, not the actual statements. Let's call such representations of proofs using dots and arrows *proof diagrams*.

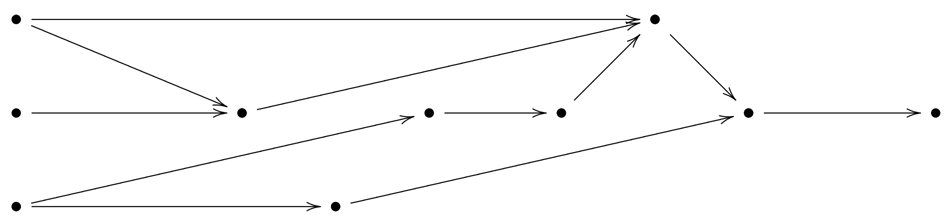

Figure 13 – A proof diagram representing Figure 3.

Exercise 2 Identify the statement in Figure 3 that corresponds to each dot in Figure 13. ∗

Not just any diagram of dots and arrows can qualify as a proof diagram. For one thing, any reasonable proof can be represented using a finite number of dots (statements) and arrows, with at least one dot and arrow. We also don't want to derive a statement from itself; that would be as silly as saying that "Today is Monday because today is Monday", which corresponds to a *loop*. The conclusion could actually be true (today might *really* be Monday), but this form of reasoning is invalid. If we had previously established "Today is Monday" to be true, the loop would be redundant; otherwise, "Today is Monday" would be uncertain and cannot be used to derive anything, let alone itself. So a proof diagram can't have a loop, that is, any part that looks like  .

Exercise 3 With reference to Figure 14, why can proof diagrams contain (b) and (d) but not (a) and (c)? Why is (b) redundant? ∗

(a) Invalid (b) Valid but redundant (c) Invalid (d) Valid

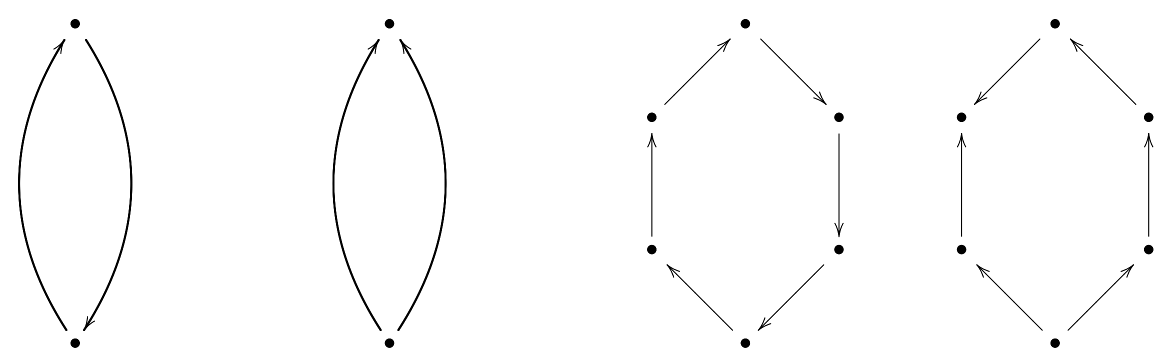

Figure 14 – Hypothetical parts of proof diagrams.

With reference to Figure 14, (a) has more than one arrow between its dots; the same holds for (b). (c) features a *directed cycle*, which is a set of arrows that allows one to travel in a circuit by following their directions. (d) features a *cycle*, which is a set of arrows that may or may not form a directed cycle, but can be turned into a directed cycle by switching the directions of some arrows.

Exercise 4 Find two cycles in Figure 3 and two cycles in Figure 11. $*$

Using this terminology we can define proof diagrams more precisely:

Definition 1 (Proof diagram) A *proof diagram* is a diagram of dots and arrows between them, without loops, directed cycles or multiple arrows between any pair of dots. The diagram must also have at least one dot and arrow. ❞

Definitions are as important in mathematics as assumptions, one reason being the material form that they give to ideas, allowing concepts to manifest in the mathematical realm within the reach of mathematical investigation—such as the following exercises.

Exercise 5 Suppose we are given a proof diagram, but not the proof that it represents. How do you identify whether each dots represents an assumption, intermediate result or conclusion? $*$

Compare the result of the last exercise to Exercise 1.

Exercise 6 Using Definition 1, explain why a proof must have at least one assumption and at least one conclusion. $*$

Exercise 7 Suppose that a certain dot of a proof diagram represents an assumption P. How is the corresponding proof affected by replacing P with $\bullet \to P$ in the proof diagram? What about replacing P with $P \to \bullet$? $*$

Exercise 8 How would the answers to Exercise 7 change if P were to be an intermediate result or conclusion instead? $*$

Exercise 9 Suppose we are tasked to "translate" a proof diagram back into its original "English-language" proof. How would we decide the order in which the dots (statements) are to be mentioned? $*$

Exercise 10 Suppose we are given a proof diagram with all arrowheads erased, so we don't know which directions the arrows used to be pointing in. How can we draw arrowheads to produce another proof diagram (not necessarily representing the original proof)? $**$

A good definition also shares a symbiotic relationship with the theory that springs from it. A fruitful partnership between definitions and theory allows mathematicians to derive clean results and conclusions that cut to the heart of the matter. For example, if Definition 1 didn't exclude directed cycles, then the exercises after it would have had to perform that exclusion by replacing every mention of "proof diagram" by "proof diagram without directed cycles", cluttering them up. In a sense, definitions have to reflect what the theory and investigations are really concerned with, which is why most math definitions we learn today are actually the "highly polished" products of a long historical process: mathematicians try out some definitions, derive a theory, then incrementally refine definitions and theory due to their mutual influence. It can take centuries to figure out the truly important consequences of an idea, as embodied in its definitions and theory; some fields at the frontier of mathematical research are still figuring it out.

Note that definitions are not assumptions; a definition is more like an entry in a dictionary, assigning a new name to a certain combination of previously understood concepts. A definition asserts nothing about existence of its subject; we could define *Santa Claus* as the man who personally delivers presents to all children on earth during Christmas, even if nobody in the world would fit that description. However, sometimes even the mere suggestion of a certain combination of ideas in a definition conveys a creative insight, an original perspective.

Exercise 11 Why were proof diagrams defined to have at least one dot and arrow? $*$

Returning to the result of Exercise 6, it seems that mathematical reasoning cannot free itself from its greatest "weakness", assumptions. Is there any knowledge that is certain—pure truth, perfectly justified without assumptions? Some deny the possibility of certain knowledge, claiming that there are only three ways to justify an argument:

Münchhausen trilemma. Every logical argument belongs to one of the following categories:

1. A *circular argument*, where derived results double back to justify the very basis used to derive them.
2. A *regressive argument*, where A is justified by B, which is justified by C, which is justified by D, and so on without end. (It's turtles all the way down!)
3. An argument that takes some bedrock assumptions for granted and justifies everything else in terms of them. ☆

Figure 15 – The Münchhausen trilemma is named after Baron Münchhausen, who in a story was said to have pulled himself—and the horse he was riding on—out of a quagmire by his own hair.

Exercise 12 Which category of the Münchhausen trilemma does a proof corresponding to a proof diagram belong to? For each of the remaining categories, draw two different diagrams of dots and arrows that represent that type of argument. *

Arguments from the first two categories are usually deemed invalid, leaving us with the third category. One way out of this dilemma—worse, a trilemma—for scientists is to accept that certain knowledge is impossible, but respond that some knowledge can be verified to be *very nearly true* if it is *falsifiable* (has a chance of being proven wrong by experiment) but no experiment has refuted it yet.[6] A solution for mathematicians is to accept that mathematical truth depends on assumptions, but that every set of assumptions opens up new worlds of study, each world offering its own beautiful truths. What do you think?

Does the Münchhausen trilemma exhaust every possibility? Are there better alternatives?

[6]Refer back to Newton's rules of reasoning.

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Image Citations

[1] Library, Cambridge University. *Page 37 of Isaac Newton's annotated copy of* Philosophiæ Naturalis Principia Mathematica. Licensed under CC BY-NC 3.0. URL: http://cudl.lib.cam.ac.uk/view/PR-ADV-B-00039-00001/97 (cited on page 9).

[2] Talifero. *Kepler's law 2 en*. I rotated it to portrait, changed the font to Utopia, made a few cleaning tweaks and added a planet and the orbit label. Licensed under CC BY-SA 2.0 Austria. URL: http://commons.wikimedia.org/wiki/File:Kepler's_law_2_en.svg (cited on page 14).

Hints to the Exercises

1 Which one of Newton's rules of reasoning would be violated by deriving an explanation from other assumptions? $*$

3 Consider the reason why loops are not allowed. $*$

5 If a dot has arrows pointing to it but none pointing away from it, what does it represent? $*$

7 Use the results of Exercise 5 to indentify the assumptions, intermediate results and conclusions after the replacement. $*$

9 If P is used to derive Q then P must be mentioned before Q. $*$

10 How can we choose the direction of each arrow to prevent the formation of directed cycles from "doubling back"? $**$